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1 Introduction

1.1 Mathematical formulation of enzyme reactions

A biochemical reaction suitable for describing a channel for molecular communications in a well-mixed environment can be described as [7]:



with the substrate S, the enzyme E, the complex C and the Product P. Assuming a well-mixed environment (i.e. the concentrations are not space-dependent), this, after some conversion, can be rewritten into a set of 2 time-dependent differential equations (with the time-dependent concentrations $c(t)$ and $s(t)$ of complex and substrate) [Rabenstein, Bersani]:

$$\dot{s}(t) = -k_1 e_T s(t) + k_{-1} c(t) + k_1 s(t) c(t) \quad (2)$$

$$\dot{c}(t) = k_1 e_T s(t) - (k_{-1} + k_2) c(t) - k_1 s(t) c(t). \quad (3)$$

1.2 Numerical Solution

Equations (2) and (3) can be solved numerically to obtain a method for graphical evaluation of the analytical solutions, which will be calculated in the following chapters. Exemplary solutions are shown in Figure 1. The characteristic behaviour of the concentrations in the case of high enzyme concentrations can be seen: The complex is produced by the reaction of enzyme and substrate, therefore the concentration increases from the initial value until the decomposition into enzyme and product becomes more dominant. The concentration then begins to decrease, asymptotically approaching zero. The substrate concentration decreases steadily in this example, as the rate constant k_1 is much bigger than k_{-1} , therefore the substrate and enzyme are steadily converted into complex.

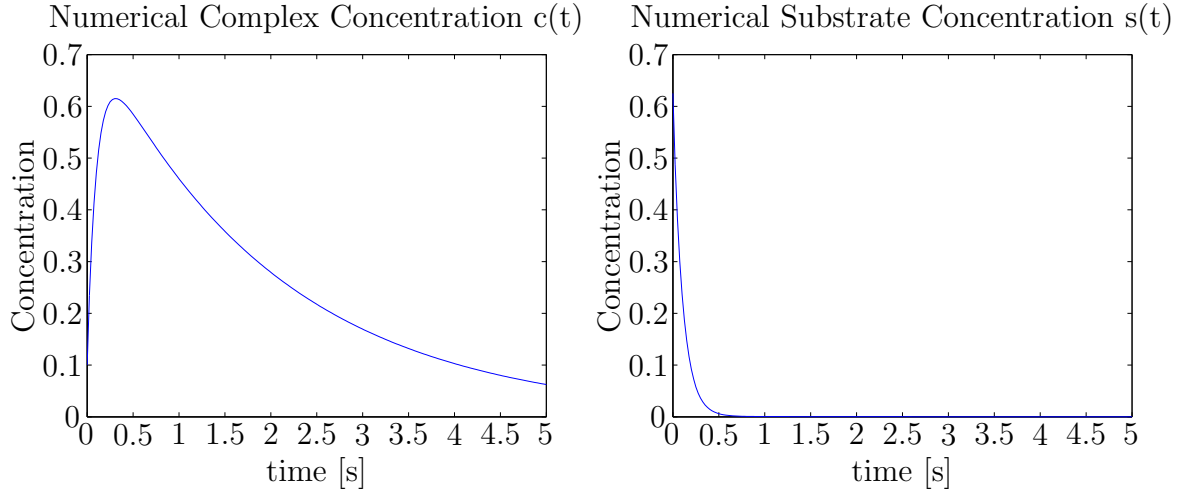


Figure 1: Exemplary numerical solution for $s_0 = 0.625$, $e_T = 10$, $c_0 = 0.1$, $k_1 = 1$, $k_{-1} = 0.01$, $k_2 = 0.5$

1.3 Amplitude Scaling

In a closed system, the total amount of enzyme e_T is constant. Therefore, the initial conditions of the system can be written as

$$s(0) = s_0; \quad e(0) = e_0; \quad c(0) = c_0; \quad e_0 + c_0 = e_T. \quad (4)$$

Now, the concentrations are amplitude-scaled to make them dimensionless

$$\mu(t) = \frac{s(t)}{s_0}; \quad \nu(t) = \frac{c(t)}{e_T}. \quad (5)$$

The scaled initial conditions are now given by:

$$\mu(0) = 1; \quad \nu(0) = \frac{c_0}{e_T} = 1 - \frac{e_0}{e_T}. \quad (6)$$

1.4 Time Scaling

Now the time constants

$$T_e = \frac{1}{k_1 e_T} \quad T_s = \frac{1}{k_1 s_0} \quad (7)$$

are introduced to allow the separation of short- and longterm behaviour of the reaction as inner and outer solutions of an asymptotic expansion [Kevorkian,Cole]. The resulting

time variables are:

$$\tau = \frac{t}{T_e} = tk_1e_T \qquad \theta = \frac{t}{T_s} = tk_1s_0. \qquad (8)$$

The relation of the time constants is defined as

$$\delta = \frac{1}{\epsilon} = \frac{T_e}{T_s}. \qquad (9)$$

For notational convenience, the dimensionless constants

$$\rho = k_{-1}T_s \qquad (10)$$

$$\lambda = k_2T_s \qquad (11)$$

$$K = \rho + \lambda \qquad (12)$$

are introduced. The time- and amplitude scaled concentrations are then defined as:

$$\sigma_e(\tau) = \mu(t) \text{ and } \gamma_e(\tau) = \nu(t) \qquad (13)$$

for the short-term behaviour and

$$\sigma_s(\theta) = \mu(t) \text{ and } \gamma_s(\theta) = \nu(t) \qquad (14)$$

for the long-term behaviour, with the derivatives

$$\sigma'_e(\tau) = T_e\dot{\mu}(t), \quad \gamma'_e(\tau) = T_e\dot{\nu}(t), \quad \sigma'_s(\theta) = T_s\dot{\mu}(t) \text{ and } \gamma'_s(\theta) = T_s\dot{\nu}(t). \qquad (15)$$

Equations 2 and 3 can now be expressed separately as

$$\sigma'_e(\tau) = f(\sigma_e(\tau), \gamma_e(\tau)), \qquad (16)$$

$$\gamma'_e(\tau) = g(\sigma_e(\tau), \gamma_e(\tau)), \qquad (17)$$

$$\sigma'_s(\theta) = f(\sigma_s(\theta), \gamma_s(\theta)) \text{ and} \qquad (18)$$

$$\gamma'_s(\theta) = g(\sigma_s(\theta), \gamma_s(\theta)). \qquad (19)$$

2 Perturbation Approach

The case of high enzyme concentration is feasible for molecular communications at the nano-scale, as it is desirable for the system to work with a low amount of information molecules [2]. Therefore, the parameter

$$\delta = \frac{s_0}{e_T} = \frac{T_e}{T_s} \quad (20)$$

can be assumed to be very small and therefore be considered as a perturbation parameter [4]. A first order perturbation approach is then applied by setting:

$$\sigma_e(\tau) = \sigma_{e,0}(\tau) + \delta\sigma_{e,1}(\tau), \quad \gamma_e(\tau) = \gamma_{e,0}(\tau) + \delta\gamma_{e,1}(\tau), \quad (21)$$

$$\sigma_s(\theta) = \sigma_{s,0}(\theta) + \delta\sigma_{s,1}(\theta) \quad \text{and} \quad \gamma_s(\theta) = \gamma_{s,0}(\theta) + \delta\gamma_{s,1}(\theta). \quad (22)$$

In the following, analytical solutions for short and long term behaviour will be calculated and matched.

2.1 Short Term Behaviour

Insertion of 21 into Eqs. 16 and 17 yields:

$$\sigma'_{e,0}(\tau) + \delta\sigma'_{e,1}(\tau) = f_{e,0}(\sigma_{e,0}(\tau), \gamma_{e,0}(\tau)) + \delta f_{e,1}(\sigma_{e,0}(\tau), \gamma_{e,0}(\tau), \sigma_{e,1}(\tau), \gamma_{e,1}(\tau)) + \mathcal{O}(\delta^2), \quad (23)$$

$$\gamma'_{e,0}(\tau) + \delta\gamma'_{e,1}(\tau) = g_{e,0}(\sigma_{e,0}(\tau), \gamma_{e,0}(\tau)) + \delta g_{e,1}(\sigma_{e,0}(\tau), \gamma_{e,0}(\tau), \sigma_{e,1}(\tau), \gamma_{e,1}(\tau)) + \mathcal{O}(\delta^2) \quad (24)$$

In the following, the dependency on τ is dropped for notational convenience. Now, the zero order terms $f_{e,0}$ $g_{e,0}$ are given by:

$$\sigma'_{e,0} = f_{e,0}(\sigma_{e,0}, \gamma_{e,0}) = -\sigma_{e,0} + \rho\gamma_{e,0} + \sigma_{e,0}\gamma_{e,0} \quad (25)$$

$$\gamma'_{e,0} = g_{e,0}(\sigma_{e,0}, \gamma_{e,0}) = 0. \quad (26)$$

The first order terms are:

$$\sigma'_{e,1} = f_{e,1}(\sigma_{e,0}, \gamma_{e,0}, \sigma_{e,1}, \gamma_{e,1}) = (\gamma_{e,0} - 1)\sigma_{e,1} + (\rho + \sigma_{e,0})\gamma_{e,1}, \quad (27)$$

$$\gamma'_{e,1} = g_{e,1}(\sigma_{e,0}, \gamma_{e,0}, \sigma_{e,1}, \gamma_{e,1}) = \sigma_{e,0}(1 - \gamma_{e,0}) - K\gamma_{e,0} = g_{e,1}(\sigma_{e,0}, \gamma_{e,0}). \quad (28)$$

Therefore, $\gamma_{e,0}$ can be directly obtained from (26), insertion into eq. (25) will then yield an analytical expression for $\sigma_{e,0}$. Then, insertion into eq. 28 and integration w.r.t. τ should give a term for $\gamma_{e,1}$, leaving the LDE eq. 27 to be solved for $\sigma_{e,1}$. This procedure is illustrated in Fig. 2.

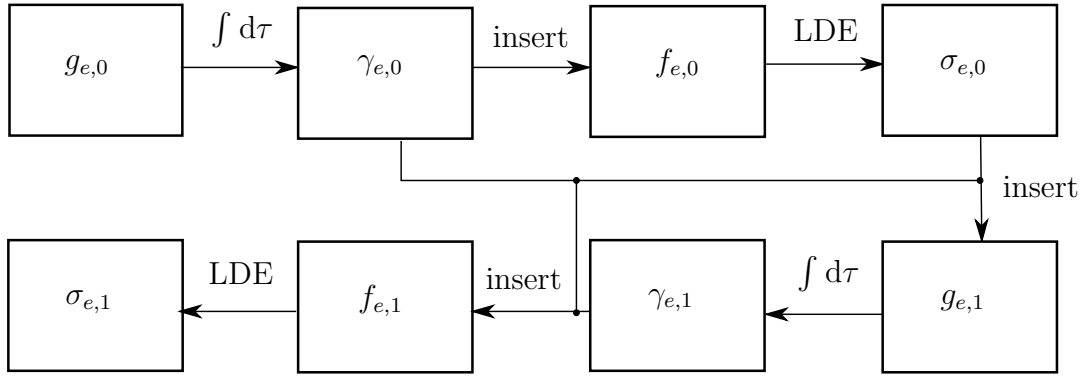


Figure 2: Obtaining the analytical expressions for the short term behaviour

Eq. 26 obviously yields:

$$\gamma_{e,0}(\tau) = \gamma_{e,0}(0) = \Gamma_{e,0}. \quad (29)$$

Insertion into eq. 25 leaves the first order LDE

$$\sigma'_{e,0}(\tau) + (1 - \Gamma_{e,0}) \sigma_{e,0}(\tau) = \rho \Gamma_{e,0}, \quad (30)$$

which can be solved for $\sigma_{e,0}(\tau)$:

$$\sigma_{e,0}(\tau) = \rho \Gamma_{e,0} \tau_0 (1 - e^{-\frac{\tau}{\tau_0}}) + \sigma_{e,0}(0) e^{-\frac{\tau}{\tau_0}} \quad (31)$$

$$= \rho(\tau_0 - 1) - \left(\rho(\tau_0 - 1) - \Sigma_{e,0} \right) e^{-\frac{\tau}{\tau_0}} \quad (32)$$

with the recurring expression

$$\tau_0 = \frac{1}{1 - \Gamma_{e,0}} \quad (33)$$

The analytical expressions (32) and (29) for the zero order terms are now inserted into (28):

$$\gamma'_{e,1}(\tau) = -\lambda\Gamma_{e,0} - e^{-\frac{\tau}{\tau_0}} \left(\rho\Gamma_{e,0} - \frac{\Sigma_{e,0}}{\tau_0} \right). \quad (34)$$

Integration w.r.t τ then yields:

$$\gamma_{e,1}(\tau) = -\lambda\Gamma_{e,0} \tau - C_{e,1} e^{-\frac{\tau}{\tau_0}} + \Gamma_{e,1}, \quad (35)$$

with the abbreviations

$$C_{e,1} = \Sigma_{e,0} - \rho\tau_0\gamma_{e,0} \quad (36)$$

$$\Gamma_{e,0} = \gamma_{e,0}(0) \quad (37)$$

and the integration constant $\Gamma_{e,1}$, which will be determined during the matching process. Now, with the analytical solutions for all other terms available, $\sigma_{e,1}$ can be determined from (30) as solution of

$$\sigma'_{e,1}(\tau) + \frac{1}{\tau_0} \sigma_{e,1}(\tau) = m(\tau), \quad (38)$$

with

$$m(\tau) = (\rho + \sigma_{e,0}(\tau)) \gamma_{e,1}(\tau) \quad (39)$$

The solution is given by (refer to appendix C for explanation):

$$\sigma_{e,1}(\tau) = e^{-\int \frac{1}{\tau_0} d\tau} \left[\int m(\tau) e^{\int \frac{1}{\tau_0} d\tau} d\tau + \Sigma_{e,1} \right], \quad (40)$$

with $\Sigma_{e,1}$ as the combined integration constant. For a comprehensive explanation of the integration method used to obtain the solution of the LDE, consider appendix C. Therefore, individual integration constants will be omitted out in the following. Now

$$\int \frac{1}{\tau_0} d\tau = \frac{\tau}{\tau_0} \quad (41)$$

Table 1: Abbreviations introduced for compact representation

$S_{e,1}$	$-\tau_0 C_{e,1}^2$
$S_{e,2}$	$-\frac{1}{2} \lambda \Gamma_{e,0} C_{e,1}$
$S_{e,3}$	$C_{e,1}(\Gamma_{e,1} - \tau_0 \rho) - \tau_0^2 \rho C_{e,1}$
$S_{e,4}$	$\tau_0^3 \rho \lambda \Gamma_{e,0}$
$S_{e,5}$	$\tau_0^2 \rho (\Gamma_{e,1} + \Gamma_{e,0} \lambda \tau_0)$

and

$$m(\tau) e^{\frac{\tau}{\tau_0}} = -\lambda \Gamma_{e,0} C_{e,1} \tau + \rho \tau_0 e^{\frac{\tau}{\tau_0}} (\Gamma_{e,1} - \lambda \Gamma_{e,0} \tau) + C_{e,1} (\rho \tau_0 + \Gamma_{e,1}) - C_{e,1}^2 e^{-\frac{\tau}{\tau_0}} \quad (42)$$

$$= -C_{e,1}^2 e^{-\frac{\tau}{\tau_0}} - \lambda \Gamma_{e,0} C_{e,1} \tau + C_{e,1} (\rho \tau_0 + \Gamma_{e,1}) + \rho \tau_0 e^{\frac{\tau}{\tau_0}} (\Gamma_{e,1} - \lambda \Gamma_{e,0} \tau) \quad (43)$$

are calculated and integrated w.r.t. τ :

$$\int m(\tau) e^{\frac{\tau}{\tau_0}} d\tau = -S_{e,1} e^{-\frac{\tau}{\tau_0}} + S_{e,2} \tau^2 + S_{e,3} \tau - S_{e,4} + S_{e,5} e^{\frac{\tau}{\tau_0}}, \quad (44)$$

with the abbreviations Inserting (44) into (40)

$$\sigma_{e,1}(\tau) = S_{e,1} e^{-2\frac{\tau}{\tau_0}} + (S_{e,2} \tau^2 + S_{e,3} \tau - S_{e,4} + \Sigma_{e,1}) e^{-\frac{\tau}{\tau_0}} + S_{e,5}. \quad (45)$$

2.2 Long Term Behaviour

Insertion of the perturbation approach into eqs. 18 and 19 gives

$$\sigma'_{s,0}(\theta) + \delta \sigma'_{s,1}(\theta) = f_{s,0}(\sigma_{s,0}(\theta), \gamma_{s,0}(\theta)) + \delta f_{s,1}(\sigma_{s,0}(\theta), \gamma_{s,0}(\theta), \sigma_{s,1}(\theta), \gamma_{s,1}(\theta)) + \mathcal{O}(\delta^2), \quad (46)$$

and

$$\gamma'_{s,0}(\theta) + \delta \gamma'_{s,1}(\theta) = g_{s,0}(\sigma_{s,0}(\theta), \gamma_{s,0}(\theta)) + \delta g_{s,1}(\sigma_{s,0}(\theta), \gamma_{s,0}(\theta), \sigma_{s,1}(\theta), \gamma_{s,1}(\theta)) + \mathcal{O}(\delta^2). \quad (47)$$

with the zero order terms

$$0 = f_{s,0}(\sigma_{s,0}, \gamma_{s,0}) = -\sigma_{s,0} + \rho\gamma_{s,0} + \sigma_{s,0}\gamma_{s,0}, \quad (48)$$

$$\gamma'_{s,0} = g_{s,0}(\sigma_{s,0}, \gamma_{s,0}) = \sigma_{s,0} - K\gamma_{s,0} - \sigma_{s,0}\gamma_{s,0}, \quad (49)$$

and the first order terms

$$\sigma'_{s,0} = f_{s,1}(\sigma_{s,0}, \gamma_{s,0}, \sigma_{s,1}, \gamma_{s,1}) = -\sigma_{s,1} + \rho\gamma_{s,1} + \sigma_{s,0}\gamma_{s,1} + \sigma_{s,1}\gamma_{s,0}, \quad (50)$$

$$\gamma'_{s,1} = g_{s,1}(\sigma_{s,0}, \gamma_{s,0}, \sigma_{s,1}, \gamma_{s,1}) = \sigma_{s,1} - K\gamma_{s,1} - \sigma_{s,0}\gamma_{s,1} - \sigma_{s,1}\gamma_{s,0}. \quad (51)$$

Therefore, $\sigma_{s,0}$ can be expressed in dependence of $\gamma_{s,0}$ through (48), addition of (48) and (49) then leads to a LDE solvable for $\gamma_{s,0}$. The sum of (51) and (52) yields another LDE with $\gamma_{s,1}$ as the only unknown variable. Finally, the expression for $\gamma_{s,1}$ is inserted into (51) and (52) to obtain $\sigma_{s,1}$. This is illustrated in fig. 3.

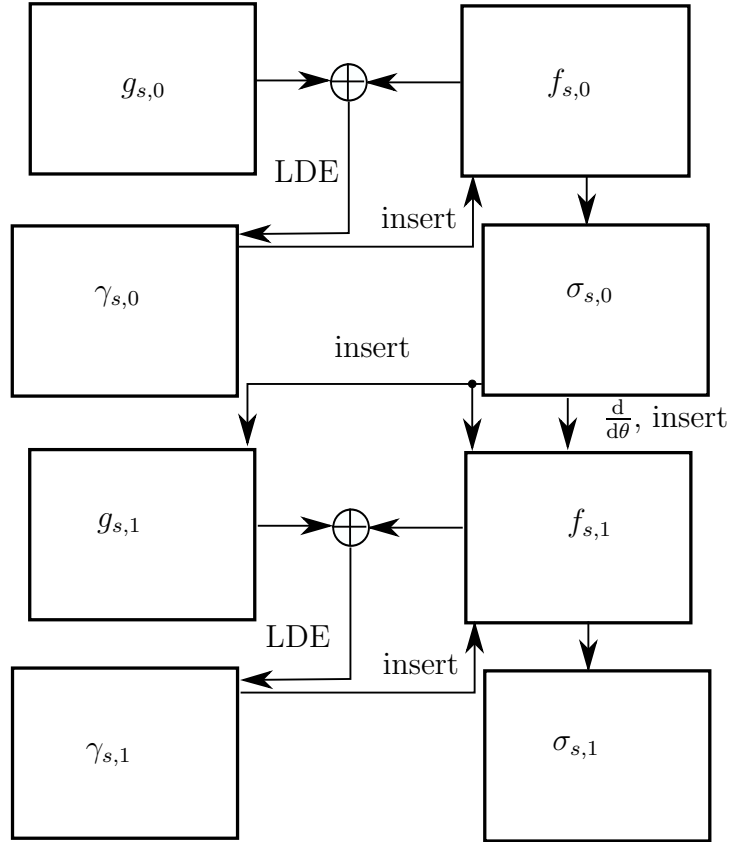


Figure 3: Obtaining the analytical expressions for the long term behaviour

The sum of (48) and (49) is:

$$\gamma'_{s,0} = -\lambda\gamma_{s,0}, \quad (52)$$

which results in the analytical expression

$$\gamma_{s,0}(\theta) = \Gamma_{s,0} e^{-\lambda\theta}, \quad (53)$$

with $\gamma_{s,0}(0) = \Gamma_{s,0}$. Insertion into (48) then yields:

$$\sigma_{s,0}(\theta) = \rho \frac{\Gamma_{s,0}}{e^{\lambda\theta} - \Gamma_{s,0}} = \rho \frac{\Gamma_{s,0}}{u(\theta)}, \quad (54)$$

with the recurring function

$$u(\theta) = e^{\lambda\theta} - \Gamma_{s,0} \quad (55)$$

The derivative of $\sigma_{s,0}$ w.r.t. θ is then given by:

$$\sigma'_{s,0}(\theta) = -\rho\lambda \frac{\Gamma_{s,0}e^{\lambda\theta}}{u(\theta)^2}. \quad (56)$$

Now, addition of eqs. 50 and 51 gives the first order LDE

$$\gamma'_{s,1}(\theta) + \lambda\gamma_{s,1}(\theta) = l(\theta), \quad (57)$$

with $l(\theta) = \rho\lambda\Gamma_{s,0}\frac{e^{\lambda\theta}}{u(\theta)^2}$. The differential equation is solvable for (appendix C gives a detailed explanation):

$$\gamma_{s,1}(\theta) = e^{-\int \lambda d\theta} \left[\int l(\theta)e^{\int \lambda d\theta} d\theta + \Gamma_{s,1} \right], \quad (58)$$

again with the combined integration constant $\Gamma_{s,1}$. With [1]

$$\int l(\theta)e^{\int \lambda d\theta} d\theta = \rho\Gamma_{s,0} \left[\ln(u(\theta) - \frac{\Gamma_{s,0}}{u(\theta)}) \right] = \rho\Gamma_{s,0}h(\theta), \quad (59)$$

the analytical expression

$$\gamma_{s,1}(\theta) = e^{-\lambda\theta} [\rho\Gamma_{s,0}h(\theta) + \Gamma_{s,1}] \quad (60)$$

is obtained. Now, $\sigma_{s,1}$ can be calculated from eq.50:

$$\sigma_{s,1} = \frac{\gamma'_{s,1} + (K + \sigma_{s,0})\gamma_{s,1}}{1 - \gamma_{s,0}}. \quad (61)$$

The derivative of $\gamma_{s,1}$ is:

$$\gamma'_{s,1}(\theta) = \lambda \left[\frac{\rho\Gamma_{s,0}}{u(\theta)} \left(1 + \frac{\Gamma_{s,0}}{u(\theta)} \right) - \gamma_{s,1}(\theta) \right] = p(\theta) - \lambda\gamma_{s,1} \quad (62)$$

With

$$\frac{(\rho + \sigma_{s,0}(\theta))\gamma_{s,1}(\theta)}{1 - \gamma_{s,0}(\theta)} = \frac{\rho(1 + \frac{\Gamma_{s,0}}{u(\theta)})(\rho\Gamma_{s,0}h(\theta) + \Gamma_{s,1})}{u(\theta)} \quad (63)$$

and

$$\frac{p(\theta)}{1 - \gamma_{s,0}(\theta)} = \frac{\rho\lambda e^{\lambda\theta}}{u(\theta)^2} \left(1 + \frac{\Gamma_{s,0}}{u(\theta)} \right), \quad (64)$$

the analytic expression for $\sigma_{s,1}$ is given by:

$$\sigma_{s,1}(\theta) = \rho \frac{1 + \frac{\Gamma_{s,0}}{u(\theta)}}{u(\theta)} \left[\rho\Gamma_{s,0}h(\theta) + \frac{\lambda e^{\lambda\theta}\Gamma_{s,0}}{u(\theta)} + \Gamma_{s,1} \right], \quad (65)$$

with the abbreviations

$$u(\theta) = e^{\lambda\theta} - \Gamma_{s,0} \quad (66)$$

$$h(\theta) = \ln(u(\theta)) - \frac{\Gamma_{s,0}}{u(\theta)} \quad (67)$$

$$p(\theta) = \lambda\rho \frac{\Gamma_{s,0}}{u(\theta)(1 + \frac{\Gamma_{s,0}}{u(\theta)})} \quad (68)$$

2.3 Matching

2.3.1 Introduction of the intermediate region

The equations in this chapter use the notation introduced in the previous chapter. Appendix D gives an overview of the relation to the notation used in [6] and [4].

The matching process is needed to assure that the inner and outer solutions fit, also, the matching conditions can be used to obtain information about unknown constants, which in this case are:

$$\Gamma_{e,0} \quad \Gamma_{s,0} \quad \Gamma_{s,1} \quad \Gamma_{e,1} \quad \Sigma_{e,0} \quad \Sigma_{e,1} \quad (69)$$

To match the short and long term solutions, an intermediate layer is introduced. In this region, both short and long term (i.e. inner and outer) solutions must be good

approximations of the true behaviour [6]. In the inner region, the time θ is of $\mathcal{O}(\delta)$, as

$$\tau = \frac{t}{T_e} = \frac{t}{T_s \delta} = \frac{\theta}{\delta}, \quad (70)$$

in the outer region, it is of $\mathcal{O}(1)$. . Now, in the intermediate region, the time is assumed to be of $\mathcal{O}(t)$, where t satisfies both [6]:

$$\lim_{\delta \downarrow 0} \frac{t}{\delta} = \infty \quad \text{and} \quad \lim_{\delta \downarrow 0} t = 0, \quad (71)$$

where \downarrow is used to describe the approximation of zero from the positive side. Now, the variable η is introduced in the intermediate region:

$$\eta = \frac{\theta}{t} = \frac{\tau \delta}{t} = \frac{1}{T_s}. \quad (72)$$

For the short and long term variables, (71) leads to

$$\lim_{\delta \downarrow 0} \tau = \infty \quad \text{and} \quad \lim_{\delta \downarrow 0} \theta = 0, \quad (73)$$

which will result in unusual notation in the following, as the behaviour w.r.t. small δ in the limit values is not clearly visible. This kind of notation, however is used in both [4] and [6] and therefore will be used for consistency.

The matching condition now requires that [6]

$$\lim_{\delta \downarrow 0} y_o(\theta) \Big|_{\theta=\eta t} = \lim_{\delta \downarrow 0} y_i(\tau) \Big|_{\tau=\eta t/\delta, \eta \text{ fixed}}, \quad (74)$$

meaning that inner and outer solution $y_i(\tau)$ and $y_o(\theta)$ have a common limit in the overlap domain.

For practical calculations, this leads to

$$\lim_{\theta \rightarrow 0} y_o(\theta) = \lim_{\tau \rightarrow \infty} y_i(\tau). \quad (75)$$

Now, the uniform approximation $y_u(\theta)$ is given by [6].

$$y_u(\theta) = y_o(\theta) + y_i\left(\frac{\theta}{\delta}\right) - \lim_{\delta \downarrow 0} y_o(\eta t), \quad (76)$$

where $\lim_{\delta \downarrow 0} y_o(\eta t)$ describes the common part of the inner and outer approximations i.e. the term that both inner and outer solutions approach in the boundary layer and which therefore would be added to the term in (76) twice, leading to an offset. The matching condition for approximations of multiple orders is given in [4] as

$$\lim_{\delta \downarrow 0} \frac{\sum_{n=0}^P y_{0,n}(\eta t)\delta^n - \sum_{n=0}^Q y_{i,n}(\eta t/\delta)\delta^n}{\delta^R} = 0. \quad (77)$$

In the given case, the inner region corresponds to a fast changing concentration in the solutions of the short term behaviour $(\sigma_e(\tau), \gamma_e(\tau))$ and the outer region corresponds to slowly approaching of a stable state in the long term behaviour $(\sigma_s(\theta), \gamma_s(\theta))$. The substrate and complex concentrations need to be matched separately. As the perturbation approach introduced in chapters 2.1 and 2.2 is first-order,

$$P = Q = 1, R = 0 \quad (78)$$

can be set in (77). This results in terms of $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ for both complex and substrate concentrations, as long as there are no dependencies on δ in the terms, as those would change the composition of orders. This means that the matching conditions yield a set of two equations for both complex and substrate concentrations, that can be used to determine unknown constants.

Now, matching conditions for the given problem are expressed as

$$\lim_{\delta \downarrow 0} (\sigma_{s,0}(\theta) + \delta\sigma_{s,1}(\theta) - \sigma_{e,0}(\theta/\delta) - \delta\sigma_{e,1}(\theta/\delta)) = 0 \quad (79)$$

for the substrate concentration and

$$\lim_{\delta \downarrow 0} (\gamma_{s,0}(\theta) + \delta\gamma_{s,1}(\theta) - \gamma_{e,0}(\theta/\delta) - \delta\gamma_{e,1}(\theta/\delta)) = 0 \quad (80)$$

for the complex concentrations. As mentioned before, both (79) and (80) should be separable into equations of $\mathcal{O}(\delta)$ and $\mathcal{O}(1)$, as long as all the terms dependent on δ show a sufficient limit behaviour, leading to a system of four equations usable to determine the remaining unknown integration constants. The uniform solutions are then given

through (76) by

$$\gamma_u(t) = \gamma_{s,0}\left(\frac{t}{T_s}\right) + \delta\gamma_{s,1}\left(\frac{t}{T_s}\right) + \gamma_{e,0}\left(\frac{t}{T_s\delta}\right) + \delta\gamma_{s,1}\left(\frac{t}{T_s\delta}\right) - \left(\lim_{\delta \downarrow 0} \gamma_{s,0}\left(\frac{t}{T_s}\right) + \delta\gamma_{s,1}\left(\frac{t}{T_s}\right)\right) \quad (81)$$

and

$$\sigma_u(t) = \sigma_{s,0}\left(\frac{t}{T_s}\right) + \delta\sigma_{s,1}\left(\frac{t}{T_s}\right) + \sigma_{e,0}\left(\frac{t}{T_s\delta}\right) + \delta\sigma_{s,1}\left(\frac{t}{T_s\delta}\right) - \left(\lim_{\delta \downarrow 0} \sigma_{s,0}\left(\frac{t}{T_s}\right) + \delta\sigma_{s,1}\left(\frac{t}{T_s}\right)\right). \quad (82)$$

In the following, the terms calculated in chapters 2.2 and 2.1 are used to assure the matching conditions are met and calculate uniform solutions. Before that, the terms used are reviewed for clarification. In (83), the composition of the inner and outer solution of the perturbation approach into zero and first order terms is shown, using the substrate concentration as an example, the same composition is of course given with the complex concentrations.

$$\begin{array}{llll} \text{inner} & \sigma_i(\tau) = & \sigma_{e,0}(\tau) & + & \delta\sigma_{e,1}(\tau) \\ \text{outer} & \sigma_o(\theta) = & \underbrace{\sigma_{e,0}(\theta)}_{\text{zero order}} & + & \underbrace{\delta\sigma_{e,1}(\theta)}_{\text{first order}} \end{array} \quad (83)$$

The matching condition (79) (or (80) for the complex concentration) can also be decomposed into $\mathcal{O}(\delta)$ and $\mathcal{O}(1)$ equations which are used for the classification of some constants in chapter 2.3.4 as shown in (84), again exemplary for the substrate concentrations. In the general case, these terms are do not correspond to the first and zero order terms in (83), because some concentration terms might cancel out or increase the order of a term so that it becomes part of the equation of another order. The terms $f_{0,1}(\theta)$ and $g_{0,1}(\tau)$ are introduced in (84) only to highlight this possible inequality.

$$\begin{array}{llll} \text{Matching Condition} & \lim_{\delta \downarrow 0} (\sigma_{e,0}(\tau) + \sigma_{s,0}(\theta) + \delta\sigma_{e,1}(\tau) + \delta\sigma_{s,1}(\theta)) & = & 0 \\ \mathcal{O}(1)\text{-Equation} & \lim_{\delta \downarrow 0} (f_0(\tau) + g_0(\theta)) & = & 0 \\ \mathcal{O}(\delta)\text{-Equation} & \lim_{\delta \downarrow 0} (f_1(\tau) + g_1(\theta)) & = & 0 \end{array} \quad (84)$$

2.3.2 Substrate Concentration

Now, the substrate concentrations are matched first. The zero and first order terms of the inner and outer solutions as obtained in chapters 2.2 and 2.1 are given in table 2.

Table 2: Terms and equation numbers for the substrate concentrations

$\sigma_{e,0}(\tau)$	$\rho\tau_0\Gamma_{e,0} (1 - e^{-\frac{\tau}{\tau_0}}) + \Sigma_{e,0} e^{-\frac{\tau}{\tau_0}}$	(32)
$\sigma_{e,1}(\tau)$	$S_{e,1}e^{-\frac{2\tau}{\tau_0}} + (S_{e,2}\tau^2 + S_{e,3}\tau - S_{e,4} + \Sigma_{e,1}) e^{-\frac{\tau}{\tau_0}} + S_{e,5}$	(45)
$\sigma_{s,0}(\theta)$	$\rho\frac{\Gamma_{s,0}}{u(\theta)}$	(54)
$\sigma_{s,1}(\theta)$	$\rho\frac{1+\frac{\Gamma_{s,0}}{u(\theta)}}{u(\theta)} \left[\rho\Gamma_{s,0}h(\theta) + \frac{\lambda e^{\lambda\theta}\Gamma_{s,0}}{u(\theta)} + \Gamma_{s,1} \right]$	(65)

In the following the relation introduced in (73) has to be kept in mind for the calculation of the limit behaviour of τ and θ for δ approaching zero.

The inner and outer solutions are given by

$$\sigma_i(\tau) = \sigma_{e,0}(\tau) + \delta\sigma_{e,1}(\tau) \quad (85)$$

$$\sigma_o(\theta) = \sigma_{s,0}(\theta) + \delta\sigma_{s,1}(\theta). \quad (86)$$

Now, the limits of the recurring functions from (66) and (67), which are needed for further calculations, are given by

$$\lim_{\delta \downarrow 0} u(\theta) \Big|_{\theta=\eta t} = 1 - \Gamma_{s,0} \quad \text{and} \quad (87)$$

$$\lim_{\delta \downarrow 0} h(\theta) \Big|_{\theta=\eta t} = \ln(1 - \Gamma_{s,0}) - \frac{\Gamma_{s,0}}{1 - \Gamma_{s,0}}. \quad (88)$$

Now, as the both first order terms $\sigma_{e,1}$ and $\sigma_{s,1}$ are multiplied with δ in (85) and (86), and they have no terms dependent on δ that would be changing the order in the evaluation of the limit values, as both the abbreviations and the exponential functions approach constants for δ approaching zero, they do not contribute to the $\mathcal{O}(1)$ equation. This means that the limit value in (76) is only dependent on the zero order terms and

the matching condition can be separately evaluated for $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ terms, which correspond to the zero and first order solutions of the perturbation approach. This means that in (84),

$$g_0(\theta) = \sigma_{s,0}(\theta) \quad g_1(\theta) = \sigma_{s,1}(\theta) \quad f_0(\tau) = \sigma_{e,0}(\tau) \quad f_1(\tau) = \sigma_{e,1}(\tau) \quad (89)$$

hold. The limit values for the zero order terms obtained from table 2 are given by

$$\lim_{\delta \downarrow 0} \left|_{\theta=\eta t} \sigma_{s,0}(\theta) = \rho \Gamma_{s,0} \lim_{\delta \downarrow 0} \left|_{\theta=\eta t} \frac{1}{u(\theta)} = \rho \frac{\Gamma_{s,0}}{1 - \Gamma_{s,0}}, \quad (90)$$

and

$$\lim_{\delta \downarrow 0} \left|_{\tau=\eta\Theta/\delta} \sigma_{e,0}(\tau) = \rho \tau_0 \Gamma_{e,0} = \rho \frac{\Gamma_{e,0}}{1 - \Gamma_{e,0}} \quad (91)$$

Evaluating the $\mathcal{O}(1)$ part of the matching condition (79) therefore yields:

$$\rho \frac{\Gamma_{e,0}}{1 - \Gamma_{e,0}} = \rho \frac{\Gamma_{s,0}}{1 - \Gamma_{s,0}}. \quad (92)$$

This relation holds if

$$\Gamma_{e,0} = \Gamma_{s,0} = \gamma_0, \quad (93)$$

so that in the following, both $\Gamma_{e,0}$ and $\Gamma_{s,0}$ will be denoted by γ_0 .

Now, a second equation of $\mathcal{O}(\delta)$, which includes the first order terms can be set up with (85) and (86):

$$\lim_{\delta \downarrow 0} \sigma_{s,1}(\theta) \Big|_{\theta=\eta\Theta} = \lim_{\delta \downarrow 0} \sigma_{e,1}(\tau) \Big|_{\tau=\eta\Theta/\delta}, \quad (94)$$

with the terms obtained from table 2

$$\lim_{\delta \downarrow 0} \sigma_{s,1}(\theta) \Big|_{\theta=\eta\Theta} = \rho \frac{1}{(1 - \gamma_0)^2} \left[\rho \gamma_0 \ln(1 - \gamma_0) - \frac{\rho \gamma_0^2}{1 - \gamma_0} + \frac{\lambda \gamma_0}{1 - \gamma_0} + \Gamma_{s,1} \right], \text{ and} \quad (95)$$

$$\lim_{\delta \downarrow 0} \sigma_{e,1}(\tau) \Big|_{\tau=\eta\Theta/\delta} = S_{e,5} = \rho \frac{1}{(\gamma_0 - 1)^2} \left[\Gamma_{e,0} - \frac{\lambda \gamma_0}{\gamma_0 - 1} \right], \quad (96)$$

which yields the following relation of the integration constants:

$$\Gamma_{e,1} = \Gamma_{s,1} + \rho \gamma_0 \left[\ln(1 - \gamma_0) - \frac{\gamma_0}{(1 - \gamma_0)} \right]. \quad (97)$$

Evaluating (82) for the functions given in (86) and (85) then yields the uniform solution:

$$\sigma_u(t) = \sigma_{s,0}\left(\frac{t}{T_s}\right) + \delta\sigma_{s,1}\left(\frac{t}{T_s}\right) + \sigma_{e,0}\left(\frac{t}{T_s\delta}\right) + \delta\sigma_{e,1}\left(\frac{t}{T_s\delta}\right) - \rho\frac{\gamma_0}{1-\gamma_0}, \quad (98)$$

with the integration constants from (69) still unknown, but the number of unknowns reduced from 6 to 4:

$$\gamma_0 \qquad \Gamma_{s,1} \qquad \Sigma_{e,0} \qquad \Sigma_{e,1} \qquad (99)$$

2.3.3 Complex Concentration

The zero and first order terms of the inner and outer solutions of the complex concentrations obtained in chapters 2.2 and 2.1 are given in table 3.

Table 3: Terms and equation numbers for the complex concentrations

$\gamma_{e,0}(\tau)$	γ_0	(29)
$\gamma_{e,1}(\tau)$	$-\lambda\gamma_0\tau - C_{e,1}e^{-\frac{\tau}{\tau_0}} + \Gamma_{e,1}$	(35)
$\gamma_{s,0}(\theta)$	$\gamma_0 e^{-\lambda\theta}$	(53)
$\gamma_{s,1}(\theta)$	$e^{-\lambda\theta}[\rho\gamma_0 h(\theta) + \Gamma_{s,1}]$	(60)

The inner and outer solutions are given by:

$$\gamma_i(\tau) = \gamma_{e,0}(\tau) + \delta\gamma_{e,1}(\tau) \quad (100)$$

$$\gamma_o(\theta) = \gamma_{s,0}(\theta) + \delta\gamma_{s,1}(\theta) \quad (101)$$

Evaluating the limit values of the zero order terms yields:

$$\lim_{\delta \downarrow 0} \gamma_{e,0}\left(\frac{t}{T_s\delta}\right) = \gamma_0 \quad (102)$$

$$\lim_{\delta \downarrow 0} \gamma_{s,0}\left(\frac{t}{T_s}\right) = \gamma_0 \quad (103)$$

The limit values for the first order terms evaluated using (71) and the expressions given in table 3 are given by

$$\lim_{\delta \downarrow 0} \gamma_{e,1}\left(\frac{t}{T_s \delta}\right) = \Gamma_{e,1} - \lim_{\delta \downarrow 0} \lambda \gamma_0 \theta = \Gamma_{e,1} \quad (104)$$

$$\lim_{\delta \downarrow 0} \gamma_{s,1}\left(\frac{t}{T_s}\right) = \ln(1 - \gamma_0) - \frac{\gamma_0}{1 - \gamma_0} + \Gamma_{s,1} = \ln(\tau_0) - \frac{\gamma_0}{\tau_0} + \Gamma_{s,1} \quad (105)$$

Note that

$$\lim_{\delta \downarrow 0} \lambda \gamma_0 \theta = 0 \quad (106)$$

due to the relation introduced in (73).

Again, the matching condition can be separately evaluated for $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ terms, which correspond to the zero and first order solutions of the perturbation approach. This means that in (84),

$$g_0(\theta) = \gamma_{s,0}(\theta) \quad g_1(\theta) = \gamma_{s,1}(\theta) \quad f_0(\tau) = \gamma_{e,0}(\tau) \quad f_1(\tau) = \gamma_{e,1}(\tau) \quad (107)$$

hold.

Both matching conditions are fulfilled if (93) (for the zero order terms) and (97) hold, also yielding additional verification for both equations, but the number of unknown constants is not reduced. With the common part determined in (102) and (103), the unified solution now is given by:

$$\gamma_u(t) = \gamma_{s,0}\left(\frac{t}{T_s}\right) + \delta \gamma_{s,1}\left(\frac{t}{T_s}\right) + \gamma_{e,0}\left(\frac{t}{T_s \delta}\right) + \delta \gamma_{e,1}\left(\frac{t}{T_s \delta}\right) - \gamma_0. \quad (108)$$

2.3.4 Boundary values

Now, as the boundary layer includes $t = 0$, the initial conditions given in (6) can be used to obtain additional information about the remaining unknown constants $\gamma_0, \Gamma_{s,1}, \Sigma_{e,1}$ and $\Sigma_{e,0}$, as they have to hold for the uniform solutions from (108) and (82):

$$\lim_{t \downarrow 0} \gamma_u(t) = \frac{c_0}{e_T} \quad \text{and} \quad (109)$$

$$\lim_{t \downarrow 0} \sigma_u(t) = 1, \quad (110)$$

As the limit value in (109) can be calculated through (108), an equation of $\mathcal{O}(\delta)$ with the first order terms and an equation of $\mathcal{O}(1)$ can be set up with the zero order terms. Similarly, (110) can be split into two equations, resulting in a set of four equations, which should be sufficient to determine the missing four constants. The distribution of the unknown constants over $\mathcal{O}(\delta)$ and $\mathcal{O}(1)$ equations resulting from the boundary conditions in is given in (109) and (110) are given in table 4.

Table 4: Distribution of unknown variables over the zero and first order boundary conditions

$\gamma_u(0), \mathcal{O}(1)$	γ_0	(82)
$\gamma_u(0), \mathcal{O}(\delta)$	$\gamma_0, \Gamma_{s,1}, \Sigma_{e,0}$	(82)
$\sigma_u(0), \mathcal{O}(1)$	$\Sigma_{e,0}$	(108)
$\sigma_u(0), \mathcal{O}(\delta)$	$\Sigma_{e,1}, \Gamma_{s,1}, \gamma_0$	(108)

Therefore, the $\mathcal{O}(1)$ -equation for the complex concentrations can be used to determine γ_0 , the $\mathcal{O}(1)$ -equation for the substrate concentrations can be used to determine $\Sigma_{e,0}$. Then the $\mathcal{O}(\delta)$ -equation for the complex concentrations can be used to determine $\Gamma_{s,1}$ and finally, the $\mathcal{O}(\delta)$ -equation for the substrate concentrations can be used to determine the last unknown constant $\Sigma_{e,1}$. The equation of $\mathcal{O}(1)$ for the concentrations concentrations is given by the zero-order part of (82), with the terms listed in table 3, with abbreviations listed in appendix A. Again, the relation introduced in (73) has to be kept in mind for the calculation of the limit behaviour of τ and θ for δ approaching zero.

$$\lim_{t \downarrow 0} \left(\gamma_{s,0}\left(\frac{t}{T_s}\right) + \gamma_{e,0}\left(\frac{t}{T_s \delta}\right) \right) - \gamma_0 = \gamma_0 = \frac{c_0}{e_T}, \quad (111)$$

yielding a solution for $\gamma_{e,0}$, which therefore will be assumed as given in the following. Now, the $\mathcal{O}(\delta)$ -Equation for the complex concentrations is given by the first order part

of (82) with the terms listed in table 3

$$\lim_{t \downarrow 0} \left(\gamma_{s,1} \left(\frac{t}{T_s} \right) + \gamma_{e,1} \left(\frac{t}{T_s \delta} \right) \right) = 0, \quad (112)$$

with

$$\lim_{t \downarrow 0} \gamma_{s,1} \left(\frac{t}{T_s} \right) = \rho \gamma_0 \left[\ln \left(\frac{1}{\tau_0} \right) - \gamma_0 \tau_0 \right] + \Gamma_{s,1} \quad (113)$$

and

$$\lim_{t \downarrow 0} \gamma_{e,1} \left(\frac{t}{T_s \Delta} \right) = -\Sigma_{e,0} + \rho \tau_0 \gamma_0 + \Gamma_{s,1} + \rho \gamma_0 \left[\ln \left(\frac{1}{\tau_0} \right) - \gamma_0 \tau_0 \right], \quad (114)$$

so that this equation contains the unknowns $\Gamma_{s,1}$ and $\Sigma_{e,0}$. The $\mathcal{O}(1)$ -Equation for the substrate concentrations is given through the zero order terms in table 2 by

$$\lim_{t \downarrow 0} \left(\sigma_{s,0} \left(\frac{t}{T_s} \right) + \sigma_{e,0} \left(\frac{t}{T_s \delta} \right) \right) - \rho \tau_0 \gamma_0 = \Sigma_{e,0} + \rho \tau_0 \gamma_0 - \rho \tau_0 \gamma_0 = 1, \quad (115)$$

yielding a solution for $\Sigma_{e,0}$, so that (112) can be solved for $\Gamma_{s,1}$

$$\Gamma_{s,1} = \frac{1}{2} \left(1 - \rho \tau_0 \gamma_0 \right) - \rho \gamma_0 \left[\ln \left(\frac{1}{\tau_0} \right) - \gamma_0 \tau_0 \right]. \quad (116)$$

A solution for $\Gamma_{e,1}$ is then given by (97)

$$\Gamma_{e,1} = \Gamma_{s,1} + \rho \gamma_0 \left(\ln \left(\frac{1}{\tau_0} \right) - \gamma_0 \tau_0 \right) = \frac{1}{2} \left(1 - \rho \tau_0 \gamma_0 \right). \quad (117)$$

The $\mathcal{O}(\delta)$ -Equation for the substrate concentrations is now given through table 2 by

$$\lim_{t \downarrow 0} \left(\sigma_{s,1} \left(\frac{t}{T_s} \right) + \sigma_{e,1} \left(\frac{t}{T_s \delta} \right) \right) = 0, \quad (118)$$

with

$$\lim_{t \downarrow 0} \sigma_{s,1} \left(\frac{t}{T_s} \right) = \rho \tau_0 \left(1 + \gamma_0 \tau_0 \right) \left[\rho \gamma_0 \left(\ln \left(\frac{1}{\tau_0} \right) - \gamma_0 \tau_0 \right) + \lambda \gamma_0 \tau_0 + \Gamma_{s,1} \right] \quad (119)$$

and

$$\lim_{t \downarrow 0} \sigma_{e,1} \left(\frac{t}{T_s \delta} \right) = \Sigma_{e,1} + S_{e,1} - S_{e,4} + S_{e,5}, \quad (120)$$

so that

$$\Sigma_{e,1} = S_{e,4} - S_{e,1} - S_{e,5} - \rho\tau_0(1 + \gamma_0\tau_0) \left[\rho\gamma_0 \left(\ln\left(\frac{1}{\tau_0}\right) - \gamma_0\tau_0 \right) + \lambda\gamma_0\tau_0 + \Gamma_{s,1} \right] \quad (121)$$

$$= S_{e,4} - S_{e,1} - S_{e,5} - \rho\tau_0(1 + \gamma_0\tau_0) \left[\frac{1}{2}(1 - \rho\tau_0\gamma_0) + \lambda\gamma_0\tau_0 \right], \quad (122)$$

which, can be simplified using the relations from appendix A to form

$$1 + \gamma_0\tau_0 = \frac{1 - \gamma_0}{1 - \gamma_0} + \frac{\gamma_0}{1 - \gamma_0} = \tau_0, \quad (123)$$

and

$$S_{e,4} - S_{e,1} - S_{e,5} = \tau_0 C_{e,1}^2 - \tau_0^2 \rho \Gamma_{e,1}, \quad (124)$$

leading to the more compact representation

$$\Sigma_{e,1} = \tau_0 [C_{e,1}^2 - \rho\tau_0(1 + (\lambda - \rho)\tau_0\gamma_0)] \quad (125)$$

All unknown constants therefore can be described by the parameters of the enzyme reaction. Table 5 lists them. The used abbreviations are given in appendix A.

Table 5: Constants defined by the boundary values and equation numbers

$\Gamma_{e,0}, \Gamma_{s,0}, \gamma_0$	$\frac{c_0}{eT}$	(111)
$\Sigma_{e,0}$	1	(115)
$\Gamma_{e,1}$	$\frac{1}{2}(1 - \rho\tau_0\gamma_0)$	(117)
$\Gamma_{s,1}$	$\frac{1}{2}(1 - \rho\tau_0\gamma_0) - \rho\gamma_0 \left[\ln\left(\frac{1}{\tau_0}\right) - \gamma_0\tau_0 \right]$	(116)
$\Sigma_{e,1}$	$\tau_0 [C_{e,1}^2 - \rho\tau_0(1 + (\lambda - \rho)\tau_0\gamma_0)]$	(125)

3 Simulation

The results were verified using Matlab by graphical comparison with the calculated numerical solution (chapter 1.2) and the perturbation approach published in [2].

Different sets of rate constants k_1, k_{-1} and k_2 , total enzyme e_T , initial complex c_0 and initial substrate s_0 were used. From them, all the introduced constants

$$\delta \quad T_e \quad T_s \quad \gamma_0 \quad \tau_0 \quad \Gamma_{e,1} \quad \Gamma_{s,1} \quad \Sigma_{e,0} \quad C_{e,1} \quad S_{e,1-5} \quad (126)$$

could be calculated (the definitions can be found in appendix A). Also, using T_e and T_s , the vector of reference times could be converted into the short and long term time scale, yielding vector representations for τ and θ which could then be used, together with the constants from (126), to calculate vector representations of

$$h(\theta) \quad u(\theta) \quad \gamma_{e,0}(\tau) \quad \gamma_{e,1}(\tau) \quad \gamma_{s,0}(\theta) \quad \gamma_{s,1}(\theta) \quad \sigma_{e,0}(\tau) \quad \sigma_{e,1}(\tau) \quad \sigma_{s,0}(\theta) \quad \sigma_{s,1}(\theta), \quad (127)$$

with the terms from (67), (66) and tables 2 and 3. Those vectors were combined to the uniform solutions from (98) and (108)

$$\gamma_u(t) \quad \sigma_u(t), \quad (128)$$

and , finally, denormalized (removing the effects of the normalization process applied in chapter 1.3)

$$s_u(t) = \sigma_u(t)s_0 \quad c_u(t) = \gamma_u(t)e_T \quad (129)$$

4 Planned Publication

The evaluation of simulation results is not contained in this document, but is to be released in [5], together with a revised version of the solutions of the perturbation approach.

A Table of Notation

k_1, k_{-1}, k_2	Reaction constants
T_e	$\frac{1}{k_1 e_T}$
T_s	$\frac{1}{k_1 s_0}$
τ	$\frac{t}{T_e} = tk_1 e_T$
θ	$\frac{t}{T_s} = tk_1 s_0$
δ	$\frac{T_e}{T_s} = \frac{s_0}{e_T}$
ρ	$k_{-1} T_s = \frac{k_{-1}}{k_1 e_T}$
λ	$k_2 T_s = \frac{k_2}{k_1 e_T}$
K	$\rho + \lambda$
$\Gamma_{e,0}$	$\gamma_{e,0}(0)$
$\Gamma_{s,0}$	$\gamma_{s,0}(0)$
τ_0	$\frac{1}{1 - \Gamma_{e,0}}$
$C_{e,1}$	$\sigma_{e,0}(0) - \rho \tau_0 \Gamma_{e,0}$
$S_{e,1}$	$-\tau_0 C_{e,1}^2$
$S_{e,2}$	$-\frac{1}{2} \lambda \Gamma_{e,0} C_{e,1}$
$S_{e,3}$	$C_{e,1}(\Gamma_{e,1} - \tau_0 \rho) - \tau_0^2 \rho C_{e,1}$
$S_{e,4}$	$\tau_0^3 \rho \lambda \Gamma_{e,0}$
$S_{e,5}$	$\tau_0^2 \rho (\Gamma_{e,1} + \Gamma_{e,0} \lambda \tau_0)$
$u(\theta)$	$e^{\lambda \theta} - \Gamma_{s,0}$
$h(\theta)$	$\ln(u(\theta)) - \frac{\Gamma_{s,0}}{u(\theta)}$
$p(\theta)$	$\lambda \rho \frac{\Gamma_{s,0}}{u(\theta)(1 + \frac{\Gamma_{s,0}}{u(\theta)})}$

B Verification of the Integral obtained from Wolfram Alpha by differentiation

The Integral used in (59) was obtained via the WolframAlpha, an online tool that can be used to obtain analytical solutions, to verify the correctness, the solution of the integral is differentiated:

$$l(\theta) = \rho\lambda\Gamma_{s,0}\frac{e^{\lambda\theta}}{u(\theta)^2} \quad (130)$$

$$k(\theta) = e^{\lambda\theta}l(\theta) \quad (131)$$

The integral $K(\theta)$ of $k(\theta)$ was given by Wolfram Alpha as:

$$K(\theta) = \int k(\theta)d\theta = \rho\Gamma_{s,0}\left[\ln(e^{\lambda\theta} - \Gamma_{s,0}) - \frac{\Gamma_{s,0}}{e^{\lambda\theta} - \Gamma_{s,0}}\right] + C. \quad (132)$$

Now,

$$\frac{dK(\theta)}{d\theta} = \rho\Gamma_{s,0}\left[\frac{\lambda e^{\lambda\theta}}{e^{\lambda\theta} - \Gamma_{s,0}} + \Gamma_{s,0}\frac{\lambda e^{\lambda\theta}}{(e^{\lambda\theta} - \Gamma_{s,0})^2}\right] \quad (133)$$

$$= \rho\Gamma_{s,0}\lambda\left[\frac{e^{\lambda\theta}(e^{\lambda\theta} - \Gamma_{s,0}) + e^{\lambda\theta}\Gamma_{s,0}}{(e^{\lambda\theta} - \Gamma_{s,0})^2}\right] \quad (134)$$

$$= \rho\lambda\Gamma_{s,0}\frac{e^{2\lambda\theta}}{u(\theta)^2} \quad (135)$$

$$= e^{\lambda\theta}l(\theta) \quad (136)$$

$$= k(\theta) \quad (137)$$

C Solution of first order LDEs

The solution of a first order linear differential equation of the form

$$y' + P(x)y = Q(x), \quad (138)$$

with the is given by [3] in chapter 9.1.1.2.5 as

$$y = e^{-\int Pdx}\left[\int Qe^{\int Pdx}dx + C\right] \quad (139)$$

. The form from (138) is given in both (38) and (57)

$$\sigma'_{e,1}(\tau) + \frac{1}{\tau_0} \sigma_{e,1}(\tau) = m(\tau) \quad (140)$$

$$\gamma'_{s,1}(\theta) + \lambda \gamma_{s,1}(\theta) = l(\theta), \quad (141)$$

D Relation of variables in Lin/Segel and Kevorkian/-Cole

The variables used in chapter 2.3.1 were unified to fit the notation of the problem at hand. Table 6 lists the equivalent variable descriptions in the respective sources.

Table 6: Relations of the variables

This report	Lin/Segel [6]	Kevorkian/Cole [4]
δ	$\epsilon, \delta(\epsilon)$	$\epsilon, \delta(\epsilon)$
t	Θ	x_η
τ	ξ	$\eta x_\eta / \epsilon$
θ	x	ηx_η
$\sigma_{e,i}(\tau), \gamma_{e,i}(\tau)$	$y_I(\xi)$	$g_i(\eta x_\eta / \epsilon)$
$\gamma_{s,i}(\theta), \gamma_{s,i}(\theta)$	$y_O(x)$	$h_i(\eta x_\eta)$

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